1988: PART A

1) $f(x) = \sqrt{x^4 - 16x^2}$

a) $x^4 - 16x^2 \geq 0$

$x^2(x^2 - 16) \geq 0$

$x^2(x+4)(x-4) \geq 0$

$f(x)$: $+$ on $x < -4$, $-$ on $-4 < x < 0$, $-$ on $0 < x < 4$, $+$ on $x > 4$ (number line marked at -4, 0, 4)

$x \leq -4 \cup x \geq 4$

b) $f(-x) = \sqrt{(-x)^4 - 16(-x)^2} = \sqrt{x^4 - 16x^2}$

$\therefore$ symmetric about the y axis

c) $f'(x) = \frac{1}{2}(x^4 - 16x^2)^{-1/2}(4x^3 - 32x)$

$= \frac{2x^3 - 16x}{\sqrt{x^4 - 16x^2}}$

d) $f'(5) = \frac{2(5^3) - 16(5)}{\sqrt{5^4 - 16(5^2)}} = \frac{170}{\sqrt{225}} = \frac{170}{15} = \frac{34}{3}$

slope $= -\frac{3}{34}$

2) $v(t) = 1 - \sin(2\pi t)$, $t \geq 0$

a) $a(t) = v'(t)$

$a(t) = -2\pi\cos(2\pi t)$

b) At rest when $v = 0$

$1 - \sin(2\pi t) = 0$

$\sin(2\pi t) = 1$

$2\pi t = \frac{\pi}{2}, \frac{5\pi}{2}$

$t = \frac{1}{4}, \frac{5}{4}$

$0 \leq t \leq 2$

$0 \leq 2\pi t \leq 4\pi$

c) $x(t) = \int v(t)\,dt$

$x(t) = \int (1 - \sin(2\pi t))\,dt$

$x(t) = t + \frac{\cos(2\pi t)}{2\pi} + C$

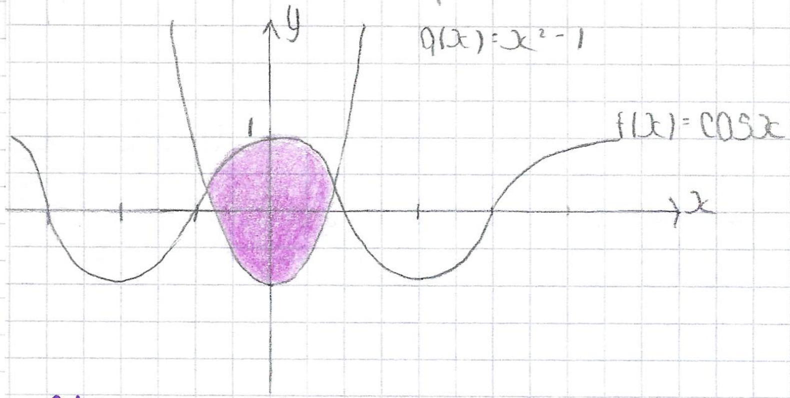
$x(0) = 0$

$0 + \frac{\cos 0}{2\pi} + C = 0$

$C = -\frac{1}{2\pi}$

$x(t) = t + \frac{\cos(2\pi t)}{2\pi} - \frac{1}{2\pi}$

5) $f(x) = \cos x$ and $g(x) = x^2 - 1$



a) $\cos x = x^2 - 1$

using G.D.C

$(1.177, 0.384)$

$(-1.177, 0.384)$

b) $\text{Area} = \int_{-1.177}^{1.177} (\cos x - (x^2 - 1))\,dx$

$= \int_{-1.177}^{1.177} (\cos x - x^2 + 1)\,dx$

$= \left[\sin x - \frac{x^3}{3} + x\right]_{-1.177}^{1.177}$

$= \sin 1.177 - \frac{1.177^3}{3} + 1.177 - \left(\sin(-1.177) - \frac{(-1.177)^3}{3} + 1.177\right)$

$= 3.114$

c)

8: PART B

4) $f(x) = 2xe^{-x}$

a) horizontal asymptote: $\lim\limits_{x\to+\infty} 2xe^{-x} = \lim\limits_{x\to+\infty} \frac{2x}{e^x} = \lim\limits_{x\to+\infty} \frac{2}{e^x} = 0^+$

$\lim\limits_{x\to-\infty} 2xe^{-x} = \lim\limits_{x\to-\infty} \frac{2x}{e^x} = 2(-\infty)e^{\infty} = -\infty$

$y = 0$

b) $f'(x) = 2e^{-x} + 2x(-1)e^{-x}$
$= 2e^{-x} - 2xe^{-x}$

$f'(x) = 0$
$2e^{-x} - 2xe^{-x} = 0$
$2e^{-x}(1-x) = 0$
$e^{-x} \neq 0$, $1 - x = 0$
$x = 1$

$f'(x)$: + | − at 1

Rel. max @ $x = 1$ since $f'(x)$ changes sign from +ve to −ve @ x

c) $f''(x) = -2e^{-x} - (2e^{-x} - 2xe^{-x})$
$= -2e^{-x} - 2e^{-x} + 2xe^{-x}$
$= -4e^{-x} + 2xe^{-x}$
$= 2e^{-x}(-2 + x)$

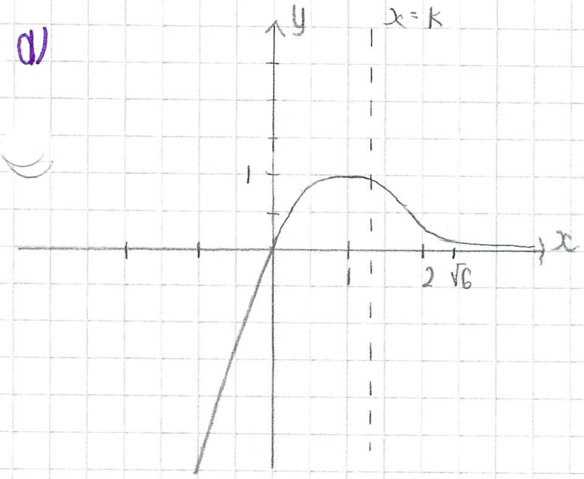
$f''(x) = 0$
$2e^{-x}(-2+x) = 0$
$e^{-x} \neq 0$, $-2 + x = 0$
$x = 2$

$f''(x)$: − | + at 2

$\cap$ $\cup$

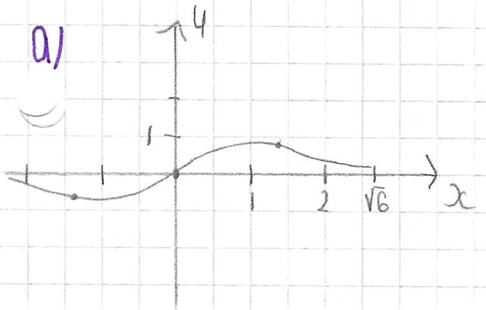
concave down on $(-\infty, 2)$

d)



5) $y = \frac{x}{x^2+2}$, $0 \leq x \leq \sqrt{6}$

a)



Vertical Asymptote: $x^2 + 2 = 0$
$x^2 \neq -2$

Horizontal Asymptote: $\lim\limits_{x\to+\infty} \frac{x}{x^2+2} = \lim\limits_{x\to+\infty} \frac{1/x}{1+2/x}$

$\lim\limits_{x\to-\infty} \frac{x}{x^2+2} = 0^-$

$y' = \frac{(x^2+2)(1) - x(2x)}{(x^2+2)^2} = \frac{-x^2+2}{(x^2+2)^2}$

$y' = 0$

$\frac{-x^2+2}{(x^2+2)^2} = 0$

$-x^2+2 = 0$, $x^2+2=0$

$x^2 = 2$ $\qquad x^2 \neq -2$

$x = \pm\sqrt{2}$

$f'(x)$: $-$ $\quad -\sqrt{2}$ $\quad +$ $\quad \sqrt{2}$ $\quad -$

$\backslash \quad \_ \quad / \quad \backslash$

Area of $R = \int_0^{\sqrt{6}} \frac{x}{x^2+2}\,dx$

$= \left[\frac{1}{2}\ln(x^2+2)\right]_0^{\sqrt{6}}$

$= \frac{1}{2}\ln 8 - \frac{1}{2}\ln 2$

$= \frac{1}{2}\ln\frac{8}{2}$

$= \frac{1}{2}\ln 4$

$= \ln 4^{1/2}$

$= \ln 2$

b) $\int_0^k \frac{x}{x^2+2}\,dx = \int_k^{\sqrt{6}} \frac{x}{x^2+2}\,dx$

$\left[\frac{1}{2}\ln(x^2+2)\right]_0^k = \left[\frac{1}{2}\ln(x^2+2)\right]_k^{\sqrt{6}}$

$\frac{1}{2}\ln(k^2+2) - \frac{1}{2}\ln 2 = \frac{1}{2}\ln 8 - \frac{1}{2}\ln(k^2+2)$

$\ln(k^2+2) = \frac{1}{2}(\ln 8 + \ln 2)$

$\ln(k^2+2) = \ln 16^{1/2}$

$k^2+2 = 4$

$k^2 = 2$

$k = \pm\sqrt{2}$

$k = \sqrt{2}$

c) $y_{AV} = \frac{1}{\sqrt{6}-0}\int_0^{\sqrt{6}} \frac{x}{x^2+2}\,dx$

$= \frac{1}{\sqrt{6}}\ln 2$

6) $f'(x) = ax^2+bx$, $f'(1) = 6$ and $f''(1) = 18$, $\int_1^2 f(x)\,dx = 18$

a) $f'(1) = 6$ $\qquad f''(x) = 2ax+b$

$a+b = 6$ $\qquad 2a+b = 18$

solving simult. $2a+b = 18$

$a+b = 6$

$a = 12$

$b = -6$

$\therefore f'(x) = 12x^2 - 6x$

$f(x) = \int (12x^2 - 6x)\,dx$

$= 4x^3 - 3x^2 + C$

$\int_1^2 (4x^3 - 3x^2 + C)\,dx = 18 \Rightarrow \left[x^4 - x^3 + Cx\right]_1^2 = 18 \Rightarrow 2^4 - 2^3 + 2C - (1 - 1 + C) = 18$

$C = 10$ $\quad \therefore f(x) = 4x^3 - 3x^2 + 1($